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Quantum Groups: A Review

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The recent development of quantum groups is summarized from the point of view of quantum physics. The emphasis is on the ideas, concepts, and motivation of these new developments.

1. INTRODUCTION

One of the most interesting recent developments in quantum physics, and mathematics, is that of a *quantum group*. This structure was developed, nearly simultaneously, in several very different ways: (a) in the statistical mechanics of Ising-type models (Jimbo, 1985, 1986) via the McGuire–Yang–Baxter equation (a solvability condition), (b) in inverse scattering theory (Sklyanin, 1982; Kulish and Reshetikhin, 1983), and (c) by mathematicians (Connes, 1985) seeking to define noncommutative differential geometry. It is our purpose in reviewing these developments primarily to explain the ideas, concepts, and particularly the motivations of this new work and only secondarily to cite the more recent achievements.

The precise definition of a quantum group (Section 2) is, for a physicist, rather formidable and inaccessible. Let us begin instead by noting that the quantum group structure combines two basic ideas: (a) the *deformation* of an algebraic structure and (b) the idea (new to quantum physics) of a (noncommutative) *comultiplication*. The idea of a deformation is not new in physics: the Poincaré group of Einsteinian relativity can be considered to be a deformation of the Galilei group of Newtonian relativity (which is recovered in the limit $c \to \infty$) and quantum mechanics can be

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considered a deformation of classical mechanics (which is recovered in the limit of $\hbar \to 0$). More precisely, the structure constants of the deformed algebra are taken to be functions of a parameter which in a fixed limit become the original structure constants. (For comultiplication see Sections 2 and 4.)

As an indication of the importance of quantum groups, let us note the following list of different fields in which this concept is currently being explored:

(a) Solvable two-dimensional systems, via inverse scattering techniques (Faddeev *et al.*, 1988; Kulish and Reshetikhin, 1989; Sklyanin, 1985; Burroughs, 1990).

(b) Solvable lattice models in statistical mechanics; anisotropic spin chain Hamiltonians (de Vega, 1989; Pasquier and Saleur, 1990; Batchelor *et al.*, 1990).

(c) Rational conformal field theory (Alvarez-Gaumé *et al.*, 1989; Moore and Reshetikhin, 1989; Gómez and Sierra, 1990).

(d) Two-dimensional gravity; three-dimensional Chern–Simons theory (Gervais, 1990*a*,*b*; Witten, 1990; Guadagnini *et al.*, 1990; Majid, 1990).

(e) Knot theory applications (Jones, 1985; Kauffman, 1990).

(f) q-Analogs to classical special functions and q-group interpretations (Gustafson, 1987; Milne, 1988; Koornwinder, 1989; Biedenharn and Lohe, 1991).

(g) Noncommutative geometry (Connes, 1985; Woronowicz, 1987, 1990; Manin, 1988).

(h) Nonstandard quantum statistics (Greenberg, 1991).

(i) Quantum Minkowski space and a q-Poincaré group (Ogievetsky *et al.*, 1991*a*,*b*).

2. DEFINITION OF A QUANTUM GROUP

The formal definition of a quantum group will be given here—but we hasten to add that the reader should probably only glance at this definition and push on to the explanations that follow!

The formal definition of a quantum group has been given by Drinfeld (1987) and by Manin (1988): *A quantum group is defined to be a (not necessarily commutative) Hopf algebra.* (Hopf algebra = bialgebra with an antipode, see below.)

2.1. Bialgebras

Let A be a k-module (k = field). Then a *bialgebra structure* on A is defined by four morphisms:

$$A \otimes A \xrightarrow{m} A \xrightarrow{\Delta} A \otimes A \quad \text{and} \quad k \xrightarrow{\eta} A \xrightarrow{\epsilon} k$$

satisfying the following axioms:

Associativity:

$$\begin{array}{ccccc} A\otimes A\otimes A & \xrightarrow{m\otimes \mathrm{id}} & A\otimes A & \xrightarrow{m} & A \\ \| & & & & \| \\ A\otimes A\otimes A & \xrightarrow{\mathrm{id}\otimes m} & A\otimes A & \xrightarrow{m} & A \end{array}$$

coassociativity:

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A\otimes A & \xrightarrow{\Delta\otimes \mathrm{id}} & A\otimes A\otimes A \\ \| & & & & \| \\ A & \xrightarrow{\Delta} & A\otimes A & \xrightarrow{\mathrm{id}\otimes \Delta} & A\otimes A\otimes A \end{array}$$

Unit:

$$\begin{array}{ccc} & A\otimes A & \\ {}^{\mathrm{id}\otimes\eta,\,\eta\otimes\mathrm{id}}\nearrow & & \searrow^{m} \\ A = A\otimes k = k\otimes A & \xrightarrow[\mathrm{id}]{} & A \end{array}$$

counit:

$$\begin{array}{ccc} & A\otimes A & \\ {}^{\Delta}\nearrow & & \searrow^{\epsilon\otimes\mathrm{id},\,\mathrm{id}\otimes\epsilon} \\ A & \xrightarrow[\mathrm{id}]{} & A = k\otimes A = A\otimes k \end{array}$$

connecting axiom ($S_{(23)}$ = exchange of 2nd and 3rd places in tensor product):

$$\begin{array}{ccc} A\otimes A \xrightarrow{m} A \xrightarrow{\Delta} A\otimes A & & \\ {\scriptstyle \Delta\otimes\Delta}\downarrow \qquad\qquad \uparrow{\scriptstyle m\otimes m} & & \\ A\otimes A\otimes A\otimes A \xrightarrow{S_{(23)}} A\otimes A\otimes A\otimes A & & \end{array}$$

2.2. Antipode

An antipode of a bialgebra (A, m, Δ) is a linear map $\gamma: A \to A$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{\Delta} & A\otimes A & \xrightarrow{\gamma\otimes\mathrm{id}} & A\otimes A & \xrightarrow{m} & A \\ \| & & & & & & \| \\ A & \xrightarrow{\epsilon} & k & \xrightarrow{\eta} & A & \xrightarrow{\mathrm{id}} & A \\ \| & & & & & & \| \\ A & \xrightarrow{\Delta} & A\otimes A & \xrightarrow[\mathrm{id}\otimes\gamma]{} & A\otimes A & \xrightarrow{m} & A \end{array}$$

3. THE QUANTUM GROUP $SU_q(2)$

The algebraic aspects of a quantum group can be most easily understood by a prototypical example: the *deformed quantal angular momentum group* $SU_q(2)$. This structure is generated by three operators, J^q_+, J^q_-, and J^q_z, which obey the commutation relations

$$[J^q_z, J^q_\pm] = \pm J^q_\pm \tag{3.1}$$

$$[J^q_+, J^q_-] = \frac{q^{J^q_z} - q^{-J^q_z}}{q^{1/2} - q^{-1/2}}, \qquad q \in \mathbb{R} \tag{3.2}$$

Remarks. (a) The commutator in (3.2) is not $2J_z$ as usual, but an *infinite* series (for generic q) involving *all odd powers*: $(J^q_z)^1, (J^q_z)^3, \ldots$. Each such power is a linearly independent operator in the *enveloping algebra*; accordingly, the Lie algebra of $SU_q(2)$ is *not* of finite dimension.

(b) For $q \to 1$, the right-hand side of equation (3.2) $\to 2J_z$. Thus we recover in the limit the usual Lie algebra. The differences noted in (a) and (b) are expressed by saying that *the quantum group $SU_q(2)$ is a deformation of the enveloping algebra of $SU(2)$*.

(c) Despite appearances, the inherent symmetry of three-space is *not* broken.

For $J^q_z \to m$, the right-hand side of equation (3.2) depends on the parameter q in a characteristic way. Let us define the *q-integer* $[n]_q$ by

$$[n]_q \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \underbrace{q^{(n-1)/2} + q^{(n-3)/2} + \cdots q^{-(n-1)/2}}_{n \text{ terms}}, \qquad n \in \mathbb{Z} \tag{3.3}$$

and

$$[n]_q! \equiv [n]_q[n-1]_q \ldots [1]_q \tag{3.4}$$

The concept of a q-*integer* is well-defined under the formal $\oplus$ *addition law*:

$$[m]_q \oplus [n]_q \equiv q^{-n/2}[m]_q + q^{m/2}[n] = [m+n]_q, \qquad m, n \in \mathbb{Z} \tag{3.5}$$

which shows the isomorphism $\{[n]_q\} \leftrightarrow \mathbb{Z}$.

Note that q-integers $[n]_q$ obey the rule $[-n]_q = -[n]_q$, with $[0]_q = 0$ and $[1]_q = 1$.

Remarks. (a) The defining relations of $SU_q(2)$ are invariant to $q \leftrightarrow q^{-1}$. Similarly, q-integers obey the symmetry $[n]_q = [n]_{q^{-1}}$. (The use of steps of *unity* for powers of q accounts for the convention using $q^{1/2}$.)

(b) The q-analogs defined by Heine (1846) did not obey the $q \leftrightarrow q^{-1}$ symmetry, and the q-factorial is related to the Heine q-analog of Euler's factorial function by $[n]_q! = q^{-n(n-1)/4}\Gamma_q(n+1)$.

4. WHY COMULTIPLICATION IS NATURAL IN QUANTUM PHYSICS

For quantum physics the new algebraic concept is the bialgebra structure, involving comultiplication. To understand this new structure, let us take an algebra A over a field k, that is, we have the following operations defined:

Multiplication:

$$A \times A \xrightarrow{m} A \tag{4.1}$$

Unit:

$$k \xrightarrow{\eta} A \qquad \text{(given by } k \to k\mathbf{1}) \tag{4.2}$$

Now adjoin *new operations* that *reverse the arrows* in (4.1) and (4.2):

Comultiplication:

$$A \xrightarrow{\Delta} A \otimes A \tag{4.3}$$

Counit:

$$A \xrightarrow{\epsilon} k \tag{4.4}$$

There are more details—as we have seen in Section 2—but in essence this structure, (4.1)–(4.4), is a Hopf algebra. This leaves open the question: What *is* comultiplication *in physical terms*?

To answer this, consider the angular momentum operator $\mathbf{J}$ in quantum mechanics. Both classically and quantum mechanically one can *add* angular momenta, that is,

$$\mathbf{J}_{\text{total}} \equiv \mathbf{J}^{(1)} + \mathbf{J}^{(2)} \tag{4.5}$$

More precisely, when we *add* angular momenta we use an action on product kets:

$$|\psi\rangle_{\text{total}} = |\psi\rangle_{(1)} \otimes |\psi\rangle_{(2)} \tag{4.6}$$

Thus the action implied by (4.5) on (4.6)—written in a more explicit and mathematically precise form—is

$$\mathbf{J}_{\text{total}} \equiv \mathbf{J}^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{J}^{(2)} \tag{4.7}$$

This action is, in effect, *a comultiplication* Δ:

$$\Delta(\mathbf{J}) \equiv \mathbf{J} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{J} \tag{4.8}$$

We conclude: *The vector addition of angular momentum in quantum physics defines a (commutative) comultiplication in a bialgebra.*

Let us now complete the definition of the quantum group $SU_q(2)$ by giving the explicit coalgebra structure. For $SU_q(2)$ the coproduct is defined by

$$\Delta(J_\pm) \equiv q^{-J_z/2} \otimes J_\pm + J_\pm \otimes q^{J_z/2} \tag{4.9}$$

$$\Delta(J_z) \equiv 1 \otimes J_z + J_z \otimes 1 \tag{4.10}$$

[Note that (4.9) breaks the $q \leftrightarrow q^{-1}$ symmetry.]

The remaining Hopf algebra operations are

$$\epsilon(1) = 1 \tag{4.11}$$

$$\epsilon(J_i^q) = 0 \tag{4.12}$$

$$\gamma(J_\pm^q) = -q^{\mp 1/2} J_\pm^q \tag{4.13}$$

$$\gamma(J_z^q) = -J_z^q \tag{4.14}$$

Remarks. (a) One of the fundamental new features of quantum groups such as $SU_q(2)$ is that one now has a *noncommutative comultiplication.* [Note that Δ in (4.9) is noncommutative.]

(b) This means tht the "addition of q-angular momenta" depends on *order*. [This in turn makes *braiding* (Kauffman, 1990) possible.]

(c) The matrices that effect the comultiplication of irreps are the *q-Wigner–Clebsch–Gordan* coefficients (Biedenharn, 1990).

5. SOME FURTHER DEVELOPMENTS IN QUANTUM GROUPS

5.1. q-Analog-Boson Operators (Biedenharn, 1989; Macfarlane, 1989; Sun and Fu, 1989)

Consider the *q-creation operator* a^q and its Hermitian conjugate the *q-destruction operator* $\bar{a}^q$. The *q-boson vacuum ket* $|0\rangle_q$ is defined by

$$\bar{a}^q|0\rangle_q \equiv 0 \tag{5.1}$$

We postulate the algebraic relation

$$\bar{a}^q a^q - q^{1/2} a^q \bar{a}^q = q^{-N_q/2} \tag{5.2}$$

where N_q is the (Hermitian) *number operator* with

$$[N_q, a^q] = a^q \tag{5.3}$$

$$[N_q, \bar{a}^q] = -\bar{a}, \qquad \text{with } N_q|0\rangle \equiv 0 \tag{5.4}$$

This algebra is a q-analog generalization of the Heisenberg algebra, which is recovered in the limit $q \to 1$.

The normalized ket vectors (orthonormal n-quanta eigenstates) have the form

$$|n\rangle_q = ([n]_q!)^{-1/2}(a^q)^n|0\rangle_q \tag{5.5}$$

where $[n]_q!$ is a q-analog of the factorial function and $(a^q)^n$ denotes the nth power of a^q.

5.2. A q-Analog of the Harmonic Oscillator

Let us define the q-momentum operator by

$$P_q \equiv i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(a_q - \bar{a}_q) \tag{5.6}$$

and the q-position operator by

$$Q_q \equiv \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a_q + \bar{a}_q) \tag{5.7}$$

The Hamiltonian for this q-oscillator is then

$$\mathscr{H}_q = \frac{\hbar\omega}{2}(\bar{a}_q a_q + a_q\bar{a}_q) \tag{5.8}$$

This q-Hamiltonian operator $\mathscr{H}_q$ is diagonal on the eigenstates $|n\rangle_q$ and has the eigenvalues

$$\mathscr{H}_q \to E_q(n) = \frac{\hbar\omega}{2}([n+1]_q + [n]_q) \tag{5.9}$$

The energy levels are no longer uniformly spaced (except for $q \to 1$). For $q \gg 1$, the energy levels become *exponential.*

The uncertainty relation for q-position and q-momentum for this q-oscillator is interesting. The uncertainty is *minimal* (and independent of n) only in the limit $q \to 1$; the uncertainty *increases* with n for $q \neq 1$. Any attempt to measure position accurately in a q-harmonic oscillator will necessarily involve large energies and a corresponding characteristic *increase in the intrinsic uncertainty.* This property and similar ones have led to suggestions (Majid) that quantum groups might be of help in controlling infinities in quantum field theories. Let us note that there are other definitions of a q-harmonic oscillator in the (now large) literature (Atakishiev and Suslov, 1991).

We remark that the q-bosons allow a q-analog to the Jordan–Schwinger map (Biedenharn, 1989). This mapping nicely leads to a determination of the q-WCG coefficients, and, in fact, to the $(q-3nj)$ coefficients (Biedenharn, 1990).

5.3. Representation Theory

For the generic case ($q \in \mathbb{R}$ or, more generally, q not a root of unity), we have the fundamental result of Lusztig (1988) and Rosso (1988) for a complex simple Lie algebra g.

Theorem. Let $\dim g < \infty$, and assume that q is generic. Then an irreducible integrable g-module can be deformed to the quantum group module $U_q(g)$. The dimensionality of each weight space is the same as in the case $q = 1$.

Physicists are interested in explicit bases for representations and we deduce (Biedenharn, 1991) from this theorem the following specific information:

Theorem. For $q \in \mathbb{R}$, a unitary $SU_q(n)$ irrep labeled by the Young frame $[m_{1n}, m_{2n}, \ldots, m_{n-1n}, 0]$ is a flag manifold whose individual vectors are labeled by a Gel'fand–Weyl pattern (m_{ij}) exactly as for the undeformed case $(q = 1)$ of $SU(n)$.

This allows us to state a general result (Biedenharn, 1991) giving the algebraic matrix elements of all q-group generators (in the Weyl–Chevalley basis) between vectors in this manifold for all $SU_q(n)$:

Theorem. Algebraic matrix elements $\langle (m') | X_\alpha | (m) \rangle$, for $(m'), (m)$ Gel'fand–Weyl patterns in $SU_q(n)$, and X_α a q-generator in the Chevalley–Weyl basis are identical to matrix elements in $SU(n)$, labeled similarly, except that each linear algebraic factor (x) is replaced by the q-number algebraic factor $[x]_q$. In particular, there are no q-factors (factors of the form q^β).

5.4. Multiparametric Deformations

Once one has learned how to define quantum groups as *one-parameter* deformations of a Lie group, it is very natural to ask about the possibility of *multiparametric* deformations. For *simple* Lie groups the answer was given by Drinfeld (1987): the one-parameter deformation is unique (to within twisting).

This result has been generalized recently by Truini and Varadarajan (1992) to *reductive Lie groups* and for these groups they have determined the most general multiparametric deformation. (A reductive Lie group has a Lie algebra which is a direct sum of a semisimple Lie algebra and an Abelian Lie algebra.) For such groups, the most general deformation is a specialization of a universal deformation having $\frac{1}{2}(N - C)(N + C - 1) + M$ parameters, where N is the rank of the reductive algebra, M is the number of simple components, and C is the dimension of the center.

6. THE VIEWPOINT OF NONCOMMUTATIVE GEOMETRY

In the discussion of quantum groups so far we have concentrated on the algebraic aspects, which—in the language of Lie groups—means that we have looked at the "infinitesimal transformations." When we consider the *finite transformations*, a novel characteristic feature of noncommutative comultiplication becomes apparent: *the coordinates of the finite transformations are noncommutative.*

To make this basic result clear, let us again consider $SU_q(2)$, and focus attention on the fundamental 2×2 irrep matrices. One finds (Nomura, 1990)

$$\mathscr{D}^{(1/2)} = \begin{pmatrix} x & u \\ v & y \end{pmatrix} \tag{6.1}$$

where the matrix elements of this irrep matrix (the coordinate functions) are noncommuting objects obeying

$$\begin{aligned} &ux = q^{1/2}xu, \qquad vx = q^{1/2}xv, \qquad yv = q^{1/2}vy \\ &yu = q^{1/2}uy, \qquad uv = vu \end{aligned} \tag{6.2}$$

In addition, we have the condition (the determinantal unimodular condition) that

$$xy - q^{-1/2}vu = yx - q^{1/2}vu = 1 \tag{6.3}$$

Let us observe that the 2×2 matrices $(J^q_\pm, J^q_z)$ which generates these finite transformations are themselves *undeformed* (this is special to the fundamental irrep and is *not true* of other irreps). The point of our observation is that the noncommutation of the finite coordinates is *not* a consequence of deformation, but entirely a property of the noncommutative comultiplication defined for $SU_q(2)$.

This noncommutativity of the coordinates of the "quantum plane"—the underlying carrier space of the 2×2 irrep—is a fundamentally new feature of a quantum group. It means that the carrier space no longer has a manifold structure, but something more general involving noncommuting coordinates.

Physicists, of course, are very familiar with the noncommuting coordinates of phase space and often—at first glance—regard the quantum group case as *déjà vu*. This is quite mistaken! For quantum physics, the miracle of quantization is that *half the coordinates of phase space*—a subset having a commuting (manifold) structure—suffice. For quantum groups there is no such easy way out: the concept of a manifold must be generalized. What this ultimately means for physics is not yet known, and is part of the attraction of quantum groups.

We conclude this brief review by noting that the Poincaré group escapes the hypothesis of the Truini–Varadarajan theorem, so that there is probably no unique q-Poincaré group. This structure is under intense study and at least one form of a q-Poincaré group (Ogievetsky *et al.*, 1991*a*,*b*) has *noncommuting* Minkowski parameters. It will be most interesting to see what this means for fundamental physics!

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